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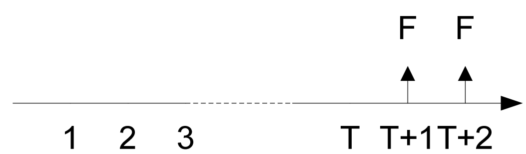
Present Value 2: Combined Forms

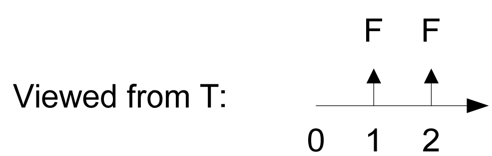
The fundamental equations for present value (see "Present Value 1: Fundamentals") can be combined to analyze more complex cash flows. Here are a couple of important special cases. In all examples, the interest rate is given by r.

Infinite stream with a delayed start

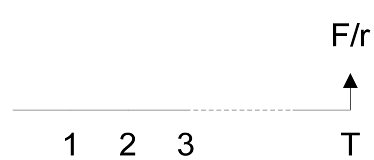
Suppose an infinite stream of equal payments of F dollars begins in year T+1. The cash flow diagram would be:



The present value can be computed in two steps. First, the infinite stream is converted to an equivalent lump sum (one-time) payment in the year before the first payment arrives—in this case, year T. The lump sum amount is what the decision maker would need to have on hand in year T in order to be able to produce the stream of F payments starting in year T+1. Computing the amount is easy because from year T's perspective, the stream of payments is very simple: the payment at T+1 occurs 1 year in the future, the payment at T+2 is 2 years in the future, and so on. Thus, from period T's perspective it's an infinite stream with payments beginning in one year in the future:



From period T's perspective, therefore, the value of that stream is F/r. Thus, the original infinite stream from T+1 on can be replaced by a single lump-sum payment of F/r in period T:



That is, someone who had F/r in year T would have exactly the right amount of money to be able to replicate the original payments of F dollars starting in T+1.

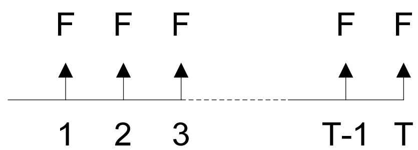
The second step is to compute the present value at year 0 of the lump sum payment in period T. In other words, step 2 determines how much the decision maker needs to have on hand in year 0 in order for it to grow to F/r in period T. That's a straightforward application of the usual present value formula but with F/r inserted for the payment at T:

$$PV = \frac{\frac{F}{r}}{(1+r)^T}$$

As an example, suppose a stream pays \$1M per year beginning in year 11, and the interest rate is 10%. In step 1, the lump sum payment in year 10 would be \$1M/0.1 or \$10M. In step 2, the final PV would be \$10M/(1.1)^10 = \$3.86M.

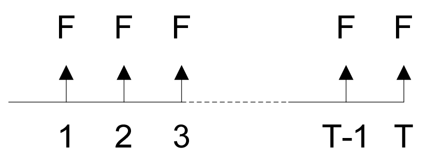
Long but finite stream

Now suppose a stream has equal payments from year 1 to T but no payments after that:



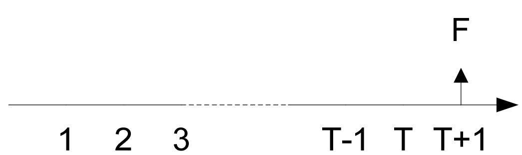
The present value could be computed by summing the present values of all T payments. However, there's a faster way that's more versatile as well. The first step is observe that the original series (call it stream "A") could be combined with one beginning at T+1 (call that stream "B") and going on forever to give an infinite series beginning at 1 (stream "C"):

Stream A:
Finite



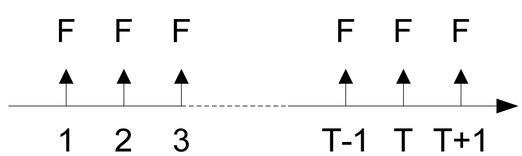
$PV_A = ?$

Stream B:
Delayed Infinite



$$PV_B = \frac{\frac{F}{r}}{(1+r)^T}$$

Stream C:
Infinite



$$PV_C = \frac{F}{r}$$

Since stream C has exactly the same payments as streams A and B combined, its present value will be the sum of the present values of the other two:

$$PV_C = PV_A + PV_B$$

Thus, the present value of stream A can be computed from the other two:

$$PV_A = PV_C - PV_B$$

Substituting in the equations for PV_C and PV_B:

$$PV_A = \frac{F}{r} - \frac{\frac{F}{r}}{(1+r)^T}$$

Factoring out F/r gives the following:

$$PV_A = \frac{F}{r}(1 - \frac{1}{(1+r)^T})$$

Essentially, this says that the present value of a long stream is equal to the present value of an infinite stream less a correction for the fact that the stream doesn't actually last forever.

For example, suppose a stream pays \$1M per year from years 1-10 and the interest rate is 10%. The value of an infinite stream would be \$1M/0.1 or \$10 million. The value of the finite stream is somewhat lower:

$$PV_A = 10M\left(1 - \frac{1}{(1.1)^{10}}\right) = 10M(1 - 0.386) = 6.14M$$

Notice that the sum of this value and the previous example total exactly \$10M, the value of receiving \$1M per year forever starting in year 1. That's because together they produce exactly that set of cash flows.

One reason this formula is particularly useful is that it can be used to solve for the amount F that would be needed to produce a series of payments that has a target present value. For example, suppose a decision maker wanted to provide 20 equal payments (in years 1-20) that would have a PV of exactly \$1 million. If the interest rate is 10%, the calculation would go as follows:

$$PV = \$1\ million$$

$$\$1\ million = \frac{F}{0.1}(1 - \frac{1}{(1.1)^{20}})$$

$$\$1\ million = 8.514 * F$$

$$F = \$117{,}459.63$$

Checking:

$$PV = \frac{117{,}459.63}{0.1}\left(1 - \frac{1}{(1.1)^{20}}\right) = \$1\ million$$