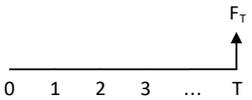
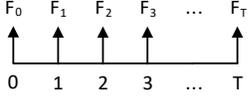
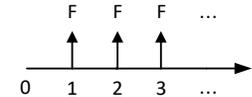
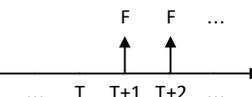
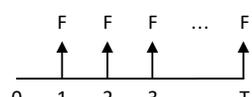
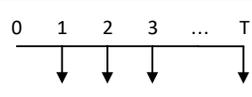


### Quick Reference Guide for Common Present Value Formulas

Several commonly-used present value formulas are shown in the table below. A discussion of each formula, including its derivation, can be found in the attached document. There are also several numerical examples showing how each one is used.

#### Summary and Table of Contents

Case	Description	Cash flow	Formula
1	Single payment in year $T$		$PV = \frac{F_t}{(1+r)^T}$
2	Stream of varying payments in years 0 to $T$		$PV = \sum_{t=0}^T \frac{F_t}{(1+r)^t}$
3	Infinite stream of identical payments beginning in year 1		$PV = \frac{F}{r}$
4	Infinite stream of identical payments with a delayed start in year $T + 1$		$PV = \frac{F/r}{(1+r)^T}$
5	Long but finite stream of identical payments in years 1 to $T$		$PV = \frac{F}{r} \left( 1 - \frac{1}{(1+r)^T} \right)$
6	Levelized costs of a project with cost $PV$ and operating in years 1 to $T$		$c = \frac{r \cdot PV}{\left( 1 - \frac{1}{(1+r)^T} \right)}$
7	Net present value		$NPV = PVB - PVC$

### Quick Reference Guide for Common Present Value Formulas

This quick reference guide summarizes several commonly-used present value formulas. On subsequent pages it shows cash flow diagrams, explanatory notes, and several numerical examples for each case. In all of the examples, the interest rate is  $r$  and year 0 is today, when the present value is being calculated. All results are rounded to the nearest dollar or thousand dollars, as appropriate.

#### Case 1: Single Payment in year $t$

A single payment of  $F_t$  dollars at time  $t$ .

*Cash flow:*



*Formula:*

$$PV = \frac{F_t}{(1 + r)^t}$$

#### Example 1.1

A payment of \$400 in year 15 when the interest rate is 7%:

$$PV = \frac{\$400}{(1 + 0.07)^{15}} = \frac{\$400}{1.07^{15}} = \$145$$

#### Example 1.2

Payment of \$150 in year 9 when the interest rate is 5%:

$$PV = \frac{\$150}{1.05^9} = \$97$$

#### Example 1.3

Payment of \$100 in year 40 when the interest rate is 3%:

$$PV = \frac{\$100}{1.03^{40}} = \$31$$

### Case 2: Stream of Varying Payments in Years 0 through T

A stream of payments over years 0 to T, where the payments may differ from one year to the next.

*Cash flow:*



*Formula:*

$$PV = \sum_{t=0}^T \frac{F_t}{(1+r)^t}$$

*Example 2.1*

A payment of \$100 in year 1 and \$400 in year 2 when the interest rate is 20%:

$$PV = \frac{\$100}{1.2} + \frac{\$400}{1.2^2} = \$361$$

*Example 2.2*

Payments of \$500 in years 2, 4 and 6 when the interest rate is 10%:

$$PV = \frac{\$500}{1.1^2} + \frac{\$500}{1.1^4} + \frac{\$500}{1.1^6} = \$1,037$$

*Example 2.3*

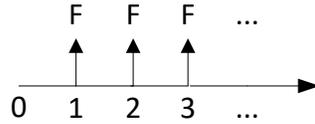
Payments of \$100 in years 5 and 10 when the interest rate is 5%:

$$PV = \frac{\$100}{1.05^5} + \frac{\$100}{1.05^{10}} = \$140$$

### Case 3: Infinite Stream of Identical Payments Beginning in Year 1

An infinite stream of identical payments  $F$  starting one year in the future (that is, no payment in year 0).

*Cash flow:*



*Formula:*

$$PV = \frac{F}{r}$$

*Example 3.1*

A payment of \$10 every year forever starting in year 1; the interest rate is 10%:

$$PV = \frac{\$10}{0.1} = \$100$$

*Example 3.2*

Payment of \$240,000 dollars starting in year 1 and continuing forever when the interest is 8%:

$$PV = \frac{\$240\text{ k}}{0.08} = \$3\text{ M}$$

*Example 3.3*

Payment of \$1000 per year starting in year 1 and continuing forever when the interest rate is 5%:

$$PV = \frac{\$1\text{ k}}{0.05} = \$20\text{ k}$$

*Why the formula works:*

There are two ways to understand the formula. The first, which isn't very intuitive, is that it's the result of applying the Case 2 formula but with the special features that there's no payment in 0, the payments after 0 are constant at  $F$ , and  $T$  goes infinity. Inserting those into the Case 2 formula and factoring out  $F$  gives the following:

$$PV = \sum_{t=0}^T \frac{F_t}{(1+r)^t} = F \sum_{t=1}^{\infty} \frac{1}{(1+r)^t}$$

That's useful because it can be shown that the sum in the revised expression converges to a very simple form:

$$\sum_{t=1}^{\infty} \frac{1}{(1+r)^t} = \frac{1}{r}$$

Replacing the sum in the previous expression with  $1/r$  gives the Case 3 formula:

$$PV = \frac{F}{r}$$

A second way to understand the formula is to rearrange it this way:

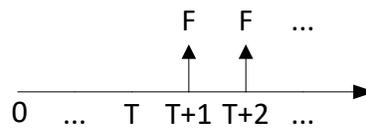
$$F = r \cdot PV$$

This shows that the present value is the amount of money that would need to be deposited in a bank account paying rate of interest  $r$  in order for the interest earned to be exactly equal to  $F$ .

**Case 4: Infinite Stream of Identical Payments with a Delayed Start in Year  $T + 1$**

An infinite stream of identical payments  $F$  that starts in year  $T + 1$  instead of year 1. No payments in years 0 through  $T$ .

*Cash flow:*



*Formula:*

$$PV = \frac{F/r}{(1+r)^T}$$

*Example 4.1*

A stream pays \$1000 per year beginning in year 11, and the interest rate is 10%. Note that the exponent in the denominator is 10, not 11:

$$PV = \frac{\$1,000}{\frac{0.1}{1.1^{10}}} = \$3,855$$

*Example 4.2*

A stream pays \$100 per year beginning in year 6 and the interest rate is 4%:

$$PV = \frac{\$100}{\frac{0.04}{1.04^5}} = \$2,055$$

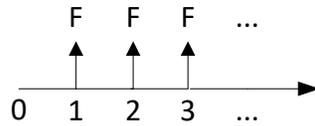
*Example 4.3*

A \$500,000 payment each year forever starting in year 21; the interest rate is 5%:

$$PV = \frac{\$500 k}{\frac{0.05}{1.05^{20}}} = \$3,769 k = \$3.769 M$$

*Why the formula works:*

The present value can be computed in two steps. First, the infinite stream is converted to an equivalent lump sum (one-time) payment in the year before the first payment arrives—in this case, year  $T$ . The lump sum amount is what the decision maker would need to have on hand in year  $T$  in order to be able to produce the stream of payments  $F$  starting in year  $T + 1$ . Computing that is easy because from year  $T$ 's perspective, the stream of payments is very simple: the payment at  $T + 1$  occurs 1 year in the future, the payment at  $T + 2$  is 2 years in the future, and so on. Thus, from period  $T$ 's perspective it's an infinite stream with payments beginning in one year in the future:



From period  $T$ 's perspective, therefore, the value of the stream can be computed using the Case 3 formula and is equal to  $F/r$ . This reasoning allows the original infinite stream from  $T + 1$  to be replaced by a single lump-sum payment of  $F/r$  in period  $T$ :



The second step is to compute the present value in year 0 of the lump sum payment in period  $T$ . That's a straightforward application of Case 1 formula but with  $F/r$  inserted for  $F_t$  and  $T$  inserted for  $t$ :

$$PV = \frac{F_t}{(1+r)^t} = \frac{F/r}{(1+r)^T}$$

### Case 5: Long but Finite Stream of Identical Payments in Years 1 to $T$

A stream of identical payments  $F$  that starts in year 1 and runs through year  $T$ . No payments from  $T + 1$  on.

*Cash flow:*



*Formula:*

$$PV = \frac{F}{r} \left( 1 - \frac{1}{(1+r)^T} \right)$$

*Example 5.1*

A stream pays \$1 million per year from years 1-20 and the interest rate is 5%:

$$PV = \frac{\$1 M}{0.05} \left( 1 - \frac{1}{1.05^{20}} \right) = \$12.462 M$$

*Example 5.2*

A stream of \$1000 per year for years 1-30 when the interest rate is 12%:

$$PV = \frac{\$1 k}{0.12} \left( 1 - \frac{1}{1.12^{30}} \right) = \$8,055$$

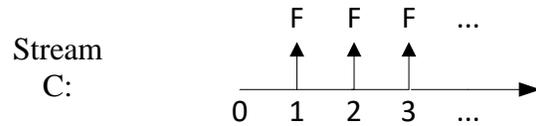
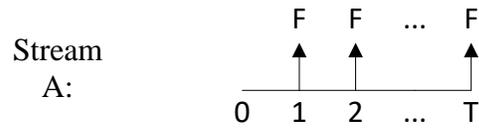
*Example 5.3*

A stream of \$500 each year for years 1-10 when the interest rate is 10%:

$$PV = \frac{\$500}{0.1} \left( 1 - \frac{1}{1.1^{10}} \right) = \$3,072$$

*Why the formula works:*

The first step is observe that the original stream (call it “A”) could be combined with one beginning at  $T + 1$  and going on forever (call that “B”) to give an infinite series beginning at 1 (call that “C”):



Since C has exactly the same payments as A and B combined, from Case 2 its present value will be the sum of the present values of the other two:

$$PV_C = PV_A + PV_B$$

However, the stream of interest is A. Rearranging to solve for its present value:

$$PV_A = PV_C - PV_B$$

Substituting in the Case 3 and Case 4 formulas for  $PV_C$  and  $PV_B$  gives:

$$PV_A = \frac{F}{r} - \frac{F/r}{(1+r)^T}$$

Factoring out  $F/r$  gives the Case 5 formula:

$$PV = \frac{F}{r} \left( 1 - \frac{1}{(1+r)^T} \right)$$

This says that the present value of a long stream is equal to the present value of an infinite stream less a correction for the fact that the stream doesn't actually last forever.

### Case 6: Levelized Cost

The “levelized” or “annualized” cost of a project is an annual payment  $F$  which would have the same present value as the project’s costs if it were paid every year over the project’s useful life. It’s a handy way to express the portion of the capital cost of a project that should be attributed to each year. It can be computed inverting the formula in the previous section to calculate the constant value  $F$  that would correspond to a given  $PV$ :

$$F = \frac{rPV}{\left(1 - \frac{1}{(1+r)^T}\right)}$$

#### Example 6.1

A project costs \$100,000 in year 0 and then operates in years 1-20. The interest rate is 5%:

$$F = \frac{0.05 * \$100,000}{\left(1 - \frac{1}{1.05^{20}}\right)} = \$8,024$$

#### Example 6.2

A project costs \$20,000 in year 0, operates in years 1-5, and is to be evaluated at an interest rate of 10%:

$$F = \frac{0.10 * \$20,000}{\left(1 - \frac{1}{1.1^5}\right)} = \$5,276$$

#### Example 6.3

A car in an organization’s vehicle fleet costs \$30,000 in year 0, is to be used for years 1-5 and then sold in year 6 for an expected resale value of \$10,000. The organization uses an interest rate of 5%. To compute the annualized cost, first compute the net present value cost of the car accounting for its resale value:

$$PV = \$30,000 - \frac{\$10,000}{1.05^6} = \$22,538$$

Then compute the annualized cost of the result:

$$F = \frac{0.05 * \$22,538}{\left(1 - \frac{1}{1.05^5}\right)} = \$5,206$$

### Case 7: Net Present Value

When a stream of payments involves costs as well as benefits then its present value is computed using the formulas above but subtracting the costs from the benefits. The result is called the net present value (NPV) of the stream to indicate that it reflects both costs and benefits.

An NPV can be calculated in two equivalent ways. One approach is to compute the present value of all the benefits, the present value of all the costs, and then to take the difference. If  $B_t$  and  $C_t$  are the benefits and costs in year  $t$ , under this approach the *NPV* would be computed as follows:

$$PVB = \sum_{t=0}^T \frac{B_t}{(1+r)^t}$$

$$PVC = \sum_{t=0}^T \frac{C_t}{(1+r)^t}$$

$$NPV = PVB - PVC$$

The other approach is to subtract each year's costs from that year's benefits and then take the present value of the stream of net payments. It would look like this:

$$NPV = \sum_{t=0}^T \frac{B_t - C_t}{(1+r)^t}$$

These two approaches give identical results and can be used interchangeably. The first form is often more intuitive but the second version is useful in some circumstances.

*Example 7.1*

A project costs \$2,000 in each of years 1, 2 and 3 and produces a benefit of \$10,000 in year 3. The interest rate is 5%. Using the first approach:

$$PVB = \frac{\$10,000}{1.05^3} = \$8,638$$

$$PVC = \frac{\$2,000}{1.05^1} + \frac{\$2,000}{1.05^2} + \frac{\$2,000}{1.05^3} = \$5,446$$

$$NPV = \$8,638 - \$5,446 = \$3,192$$

Using the second approach:

$$NPV = \frac{\$0 - \$2,000}{1.05^1} + \frac{\$0 - \$2,000}{1.05^2} + \frac{\$10,000 - \$2,000}{1.05^3} = \$3,192$$

*Example 7.2*

A project produces a \$500 benefit in year 0 and has \$300 costs in each of years 1 and 2. The interest rate is 10%. Using the first approach:

$$PVB = \frac{\$500}{1.1^0} = \$500$$

$$PVC = \frac{\$300}{1.1^1} + \frac{\$300}{1.1^2} = \$521$$

$$NPV = \$500 - \$521 = -\$21$$

Using the second approach:

$$NPV = \frac{\$500 - \$0}{1.1^0} + \frac{\$0 - \$300}{1.1^1} + \frac{\$0 - \$300}{1.1^2} = -\$21$$

*Example 7.3*

A project costs \$1,000 in years 1 through 10 and then starting in year 11 it produces benefits of \$1,200 per year forever. The interest rate is 6%. Using the first approach and applying the Case 4 formula for benefits and the Case 5 formula for costs:

$$PVB = \frac{\$1,200}{\frac{0.06}{1.06^{10}}} = \$11,168$$

$$PVC = \frac{\$1,000}{0.06} \left(1 - \frac{1}{1.06^{10}}\right) = \$7,360$$

$$NPV = \$11,168 - \$7,360 = \$3,808$$

For this problem—where the periods of benefits and costs don't overlap and the benefits go on forever—the second approach is more work and less intuitive so it is omitted.

*Example 7.4*

A project provides benefits of \$500 per year in years 1 through 5 and then starting in year 6 it has costs of \$200 per year forever. The interest rate is 10%. Using the first approach and applying the Case 5 formula for benefits and the Case 4 formula for costs:

$$PVB = \frac{\$500}{0.1} \left(1 - \frac{1}{1.1^5}\right) = \$1,895$$

$$PVC = \frac{\$200}{\frac{0.1}{1.1^5}} = \$1,242$$

$$NPV = \$1,895 - 1,242 = \$653$$

As in the previous example, the second approach is less intuitive for this case and is omitted.