Formula 3: Infinite Stream of Identical Payments

Special stream: **identical payments** starting **one year in the future**

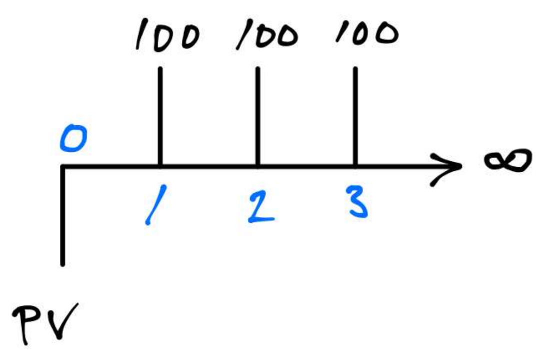
Known as an *annuity*, *perpetuity* or *consol*

- Basic idea behind endowments

Example:

Want \$100/year forever starting in 1 when $r = 10\%$
How much to deposit today?

Cash flow:



Present value of the stream:

$$PV = \frac{100}{1.1} + \frac{100}{(1.1)^2} + \frac{100}{(1.1)^3} + \cdots$$

$$PV = \sum_{t=1}^{\infty} \frac{100}{(1.1)^t}$$

- End T is infinity
- Start t is 1
- No subscript on \$100 payment

Factor out the 100 since the payments are identical:

$$PV = 100\left(\frac{1}{1.1} + \frac{1}{(1.1)^2} + \frac{1}{(1.1)^3} + \cdots\right)$$

$$PV = 100\left(\sum_{t=1}^{\infty} \frac{1}{(1.1)^t}\right)$$

Can show that the sum in parentheses converges:

$$\sum_{t=1}^{\infty} \frac{1}{(1.1)^t} = \frac{1}{0.1}$$

Substituting into the PV equation:

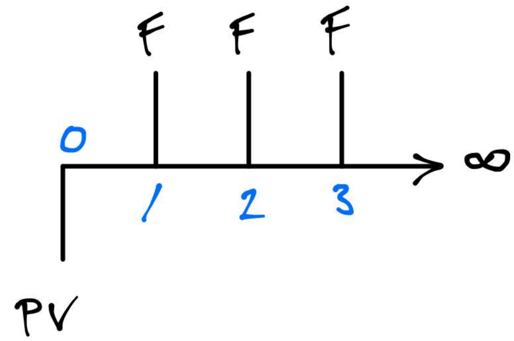
$$PV = 100\left(\frac{1}{0.1}\right) = \frac{100}{0.1}$$

$$PV = 1000$$

Common sense result:

Deposit \$1000 at 0 and withdraw \$100 interest each year

Generalizing:



$$PV = \sum_{t=1}^{\infty} \frac{F}{(1+r)^t}$$

$$PV = F\left(\sum_{t=1}^{\infty} \frac{1}{(1+r)^t}\right) \quad \text{and} \quad \sum_{t=1}^{\infty} \frac{1}{(1+r)^t} = \frac{1}{r}$$

$$PV = F\left(\frac{1}{r}\right)$$

$$PV = \frac{F}{r}$$

⚠ PV formula 3

Rearrange for handy interpretation:

$$r * PV = F$$

- PV = deposit in bank
- Left side = interest on PV
- Right side = desired withdrawal F

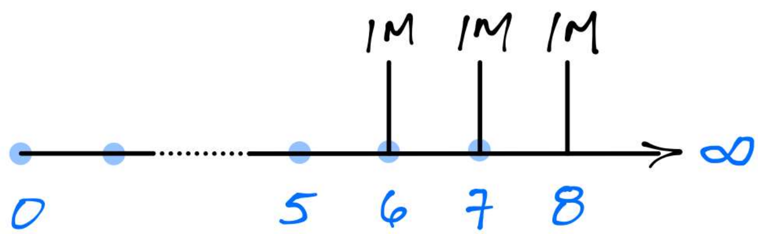
Formula 4: Infinite Stream with a Delayed Start

Special stream 2:

Identical payments starting **more than 1 period** in future

Example:

Policy produces **$1M/year** starting in **6**
No payments in **0-5**
$r = 10\%$



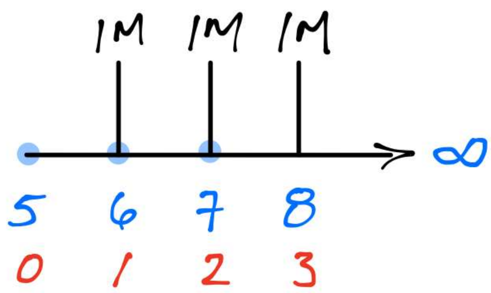
Find the PV in two steps working backward:

1. Find year-5 deposit, x_5, that would produce $1M a year from 6 on
2. Find year-0 deposit, x_0, that produces x_5 in year 5

x_0 will be the PV

Step 1:

Zoom in on payments after year 5:



blue: years from 0
red: years from 5

First payment is one year beyond year 5

Can use the infinite stream formula **in year 5**:

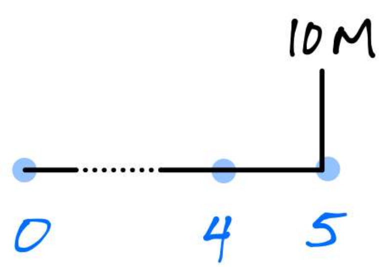
$$x_5 = \frac{F}{r}$$

$$x_5 = \frac{1\ M}{0.1} = 10\ M$$

Having \$10 M in bank **in year 5** can produce \$1 M from 6 onward

Step 2:

The \$10 M *doesn't have to be in the bank until year 5:*



Need to find deposit needed at 0

Can use the single payment formula:

$$x_0 = \frac{10\ M}{1.1^5} = \frac{x_5}{(1+r)^5}$$

$$x_0 = 6.209\ M$$

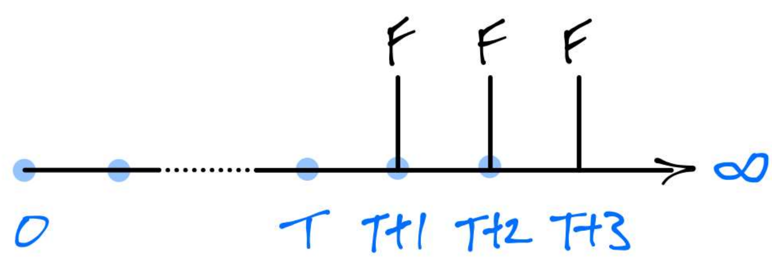
Done!

Since x_0 is the deposit at 0, it's the PV

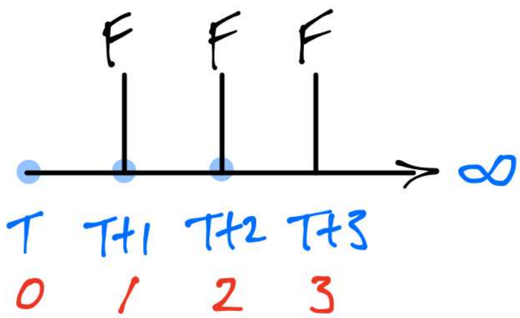
$$PV = x_0 = 6.209\ M$$

Generalizing:

Policy produces F every year starting at $\boldsymbol{T+1}$
Interest rate r

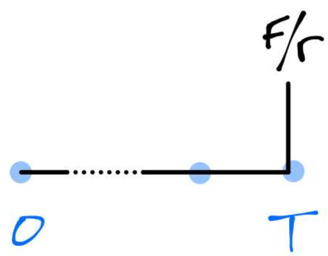


Step 1: Find x_T just before payments start



$$x_T = \frac{F}{r}$$

Step 2: Find the overall PV, x_0



$$x_0 = \frac{x_T}{(1+r)^T}$$

$$x_0 = \frac{\frac{F}{r}}{(1+r)^T}$$

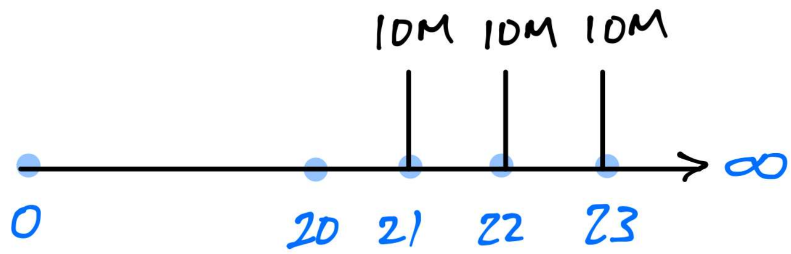
General formula:

$$\boxed{PV = \frac{\frac{F}{r}}{(1+r)^T}}$$

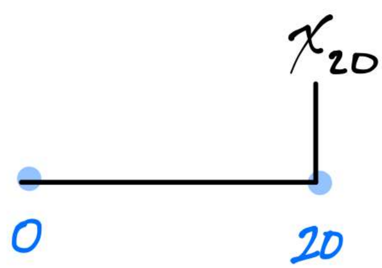
⚠ PV formula 4

Example:

Change zoning to reduce future damage from sea level rise
Gain will be **$10 M/year starting in year 21**
$r = 5\%$



Two step approach:



$$x_{20} = \frac{F}{r} = \frac{10\ M}{0.05} = 200\ M$$

$$x_0 = \frac{x_{20}}{(1+r)^{20}} = \frac{200\ M}{1.05^{20}} = 75.378\ M$$

$$PV = 75.378\ M$$

Combined formula approach:

$$PV = \frac{\frac{F}{r}}{(1+r)^T} = \frac{\frac{10\ M}{0.05}}{1.05^{20}} = 75.378\ M$$

$$PV = 75.378\ M$$